Basics of Optimization

Andrea Natale

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1 Introduction

Given a nonempty set X and a function $F : X \to \mathbb{R} \cup \{+\infty\}$, we are interested in the minimization problem

$$\inf_{x \in X} F(x).$$

The main questions that we care about are the following:

- **Existence of minimizers**: is the infimum achieved?
- **Uniqueness**: is it achieved by a unique point x^*?
- **Necessary and sufficient conditions**: can we characterize minimizers?
- **Computation**: can we design an algorithm to compute minimizers?

In this course, we will provide some (partial) answers to these questions for both finite-dimensional and infinite-dimensional problems. Why considering infinite-dimensional problems? This is illustrated by the following examples.

1.1 Rudin–Osher–Fatemi (ROF) denoising

Let $u_0 : \Omega \to [0,1]$ be a noisy image, with $\Omega \subset \mathbb{R}^2$. A possible way to remove noise from u_0 and recover a denoised image is to solve a minimization problem that we build ad-hoc for this task. A standard variational model is the so-called ROF model: for a given $p \geq 1$ and $\lambda > 0$, one solves

$$\inf_u \left\{ \lambda \int_\Omega |u - u_0|^2 \, dx + \int_\Omega |\nabla u|^p \, dx \right\}.$$

The first term is the *data-attachement* term and keeps the reconstructed image close to the noisy one u_0. The second term serves as a *regularization* and enforces smoothness of the reconstructed image. The case $p = 1$ has particularly favorable properties for image reconstruction as it can be shown to preserve sharp edges.

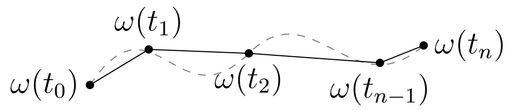
In practice the image is discrete, but the continuous formulation is useful because it reveals the analytic structure of the problem and leads to discretizations whose design is conceptually independent of the pixel count.

1.2 Geodesics

Consider two points x_0, x_1 on a set or manifold $M \subset \mathbb{R}^d$. We seek curves of minimal length connecting them.

Definition 1.1. *A* ***curve*** *on M is a continuous function $\omega : [0,1] \to M$. Its length is defined as follows:*

$$\mathrm{Length}(\omega) := \sup \left\{ \sum_{k=0}^{n-1} |\omega(t_{k+1}) - \omega(t_k)| : \begin{array}{c} n \geq 1, \\ 0 = t_0 < t_1 < \cdots < t_n = 1 \end{array} \right\}.$$



Remark 1.1 (Reparametrization invariance)**.** *If $p : [0,1] \to [0,1]$ is an increasing homeomorphism and $\widetilde{\omega} = \omega \circ p$, then*

$$\mathrm{Length}(\widetilde{\omega}) = \mathrm{Length}(\omega).$$

Thus length depends on the geometric curve, not on its parametrization.

Proposition 1.1. *If $\omega \in C^1([0,1]; \mathbb{R}^d)$, then*

$$\mathrm{Length}(\omega) = \int_0^1 |\omega'(t)| \, dt.$$

Proof. For any partition $0 = t_0 < \cdots < t_n = 1$,

$$\begin{aligned} \sum_{k=0}^{n-1} |\omega(t_{k+1}) - \omega(t_k)| &= \sum_{k=0}^{n-1} \left| \int_{t_k}^{t_{k+1}} \omega'(t) \, dt \right| \\ &\leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\omega'(t)| \, dt = \int_0^1 |\omega'(t)| \, dt. \end{aligned}$$

Taking the supremum gives $\mathrm{Length}(\omega) \leq \int_0^1 |\omega'|$.

For the converse, take the uniform partition $t_k = k/n$. Since ω' is uniformly continuous,

$$\frac{\omega(t_{k+1}) - \omega(t_k)}{1/n} \to \omega'(t_k)$$

uniformly in k as $n \to \infty$. Hence

$$\sum_{k=0}^{n-1} |\omega(t_{k+1}) - \omega(t_k)| = \frac{1}{n} \sum_{k=0}^{n-1} |\omega'(t_k)| + o(1).$$

The right-hand side converges to $\int_0^1 |\omega'(t)| \, dt$ by the Riemann-sum theorem. Therefore $\mathrm{Length}(\omega) \geq \int_0^1 |\omega'|$, proving equality. $\square$

Remark 1.2 (Jensen's inequality)**.** *If $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex and g is integrable on a finite-measure interval, then*

$$\phi\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right) \le \frac{1}{b-a}\int_a^b \phi(g(t))\,dt.$$

The finite-dimensional form is

$$\phi\left(\sum_{i=1}^N \alpha_i y_i\right) \le \sum_{i=1}^N \alpha_i \phi(y_i), \qquad \alpha_i \ge 0, \quad \sum_i \alpha_i = 1.$$

Given the characterization of length in the previous proposition, we can formulate the **geodesic problem** as follows: Given $x_0, x_1 \in M$,

$$\inf\left\{\mathrm{Length}(\omega) : \omega \in C^1([0,1];M),\ \omega(0) = x_0,\ \omega(1) = x_1\right\}.$$

Remark 1.3. *This is a problem in the* calculus of variations*: the unknown is a function, and the direct method of analysis becomes a central existence tool.*

2 Existence of minimizers

Abstract problem: Let (X, d) be a metric space, let $A \subset X$ be nonempty, and let $F : A \to \mathbb{R} \cup \{+\infty\}$. We study

$$\inf_{x \in A} F(x).$$

Remark 2.1. *Often X will be a normed vector space, and in infinite-dimensional problems we will frequently work in a Banach or Hilbert space.*

Examples of Banach spaces

- $\mathbb{R}^d$ with any norm, for example the Euclidean norm.
- $C(\overline{\Omega})$, for $\overline{\Omega}$ compact, with $\|f\|_\infty = \max_{x\in\overline{\Omega}} |f(x)|$.
- $L^p(\Omega)$, $1 \le p \le \infty$, with the usual L^p norm.
- Sobolev spaces $W^{1,p}(\Omega)$ (for suitable domains), consisting of functions whose weak first derivatives belong to L^p.

Definition 2.1 (Global minimizer)**.** *An element $x^* \in A$ is a* ***global minimizer*** *if*

$$F(x^*) \le F(x) \qquad \forall x \in A.$$

Equivalently, $F(x^*) = \inf_A F$.

Definition 2.2 (Local minimizer)**.** *An element $x^* \in A$ is a* ***local minimizer*** *if there exists $\delta > 0$ such that*

$$F(x^*) \le F(x) \qquad \text{for all } x \in A \text{ with } d(x, x^*) \le \delta.$$

A minimizer is strict *if the inequality is strict for every competing point $x \ne x^*$ in the relevant set.*

Definition 2.3 (Minimizing sequence). *A sequence $(x_n)_n \subset A$ is a **minimizing sequence** if*

$$F(x_n) \longrightarrow \inf_{x \in A} F(x).$$

Remark 2.2. *A minimizing sequence always exists whenever $A \neq \emptyset$ and $\inf_A F < +\infty$. Indeed, by the definition of the infimum, for each n one can choose $x_n \in A$ such that*

$$F(x_n) \le \inf_A F + \frac{1}{n}.$$

2.1 Existence in finite dimensions

Proposition 2.1. *Let $K \subset \mathbb{R}^d$ be nonempty and compact, and let $f : K \to \mathbb{R}$ be continuous. Then f attains both its minimum and its maximum on K.*

Proof. Let $(x_n)_n$ be a minimizing sequence. Since K is compact, it is sequentially compact, so after passing to a subsequence, $x_{n_k} \to \bar{x} \in K$. By continuity,

$$f(\bar{x}) = \lim_{k \to \infty} f(x_{n_k}) = \inf_K f.$$

Thus $\bar{x}$ is a minimizer. Apply the same argument to $-f$ to obtain a maximizer. $\square$

2.2 Lower semicontinuity

Definition 2.4 (Lower semicontinuity). *Let (X, d) be a metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$. We say that f is **lower semicontinuous (l.s.c.)** at $a \in X$ if*

$$\forall \varepsilon > 0 \ \exists r > 0 \ \text{such that } d(x, a) < r \implies f(x) \ge f(a) - \varepsilon.$$

We say that f is l.s.c. if it is l.s.c. at every point.

Remark 2.3. *Continuity additionally requires the upper estimate $f(x) \le f(a) + \varepsilon$ for x sufficiently close to a.*

Remark 2.4 (Indicator functions). *For a set $A \subset X$, define*

$$\iota_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A. \end{cases}$$

Then for a finite-valued f,

$$\inf_{x \in A} f(x) = \inf_{x \in X} \big(f(x) + \iota_A(x) \big).$$

This turns a constrained problem into an unconstrained problem for an extended-valued functional. Moreover, ι_A is l.s.c. exactly when A is closed.

Definition 2.5 (Sequential lower semicontinuity). *We say that f is **sequentially l.s.c.** at a if for every sequence $x_n \to a$,*

$$f(a) \le \liminf_{n \to \infty} f(x_n).$$

Proposition 2.2. *In a metric space, f is l.s.c. at a if and only if it is sequentially l.s.c. at a.*

Proof. If f is l.s.c. and $x_n \to a$, then for every $\varepsilon > 0$ we have $f(x_n) \geq f(a) - \varepsilon$ for all sufficiently large n. Hence $\liminf_n f(x_n) \geq f(a) - \varepsilon$. Letting $\varepsilon \downarrow 0$ gives the claim.

Conversely, suppose f is not l.s.c. at a. Then there exists $\varepsilon > 0$ such that for every n one can choose x_n with $d(x_n, a) < 1/n$ and $f(x_n) < f(a) - \varepsilon$. Thus $x_n \to a$ but $\liminf_n f(x_n) \leq f(a) - \varepsilon$, contradicting sequential l.s.c. $\square$

Example 2.1. *The function*

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0, \end{cases}$$

is l.s.c. at every point: at 0 the value $f(0) = 1$ is not exceeded from below by nearby values, although f is not continuous there.

Definition 2.6 (Epigraph)**.** *For $f : X \to \mathbb{R} \cup \{+\infty\}$, its* ***epigraph*** *is*

$$\operatorname{epi}(f) := \{(x, \lambda) \in X \times \mathbb{R} : \; f(x) \leq \lambda\}.$$

Proposition 2.3. *A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. if and only if $\operatorname{epi}(f)$ is closed in $X \times \mathbb{R}$.*

Proof. Suppose first that f is l.s.c. and $(x_n, \lambda_n) \in \operatorname{epi}(f)$ with $(x_n, \lambda_n) \to (\bar{x}, \bar{\lambda})$. Then $f(x_n) \leq \lambda_n$, hence

$$f(\bar{x}) \leq \liminf_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} \lambda_n = \bar{\lambda}.$$

Thus $(\bar{x}, \bar{\lambda}) \in \operatorname{epi}(f)$.

Conversely, assume $\operatorname{epi}(f)$ is closed and let $x_n \to \bar{x}$. Set $\alpha = \liminf_n f(x_n)$. If $\alpha = +\infty$, there is nothing to prove. Otherwise choose a subsequence x_{n_k} such that $f(x_{n_k}) \to \alpha$. Then

$$(x_{n_k}, f(x_{n_k})) \in \operatorname{epi}(f), \qquad (x_{n_k}, f(x_{n_k})) \to (\bar{x}, \alpha).$$

Closedness gives $(\bar{x}, \alpha) \in \operatorname{epi}(f)$, hence $f(\bar{x}) \leq \alpha$. $\square$

Example 2.2. *Let*

$$f(x) = \begin{cases} |x|, & x \neq 0, \\ 5, & x = 0. \end{cases}$$

Then f is not l.s.c. at 0. Moreover, $\inf_{\mathbb{R}} f = 0$, but the infimum is not attained.

2.3 The direct method in finite dimensions

Theorem 2.1 (Weierstrass, extended-valued form)**.** *Let (X, d) be a compact metric space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be l.s.c. Assume f is proper, i.e. $f(x) < +\infty$ for at least one $x \in X$. Then f admits a global minimizer.*

Proof. Take a minimizing sequence $(x_n)_n$. By compactness, after passing to a subsequence, $x_{n_k} \to \bar{x} \in X$. By lower semicontinuity,

$$f(\bar{x}) \le \liminf_{k\to\infty} f(x_{n_k}) = \inf_X f.$$

The reverse inequality is automatic from the definition of the infimum, so $f(\bar{x}) = \inf_X f$. $\square$

Definition 2.7 (Coercivity)**.** *Let (X, d) be an unbounded metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$. We say that f is* ***coercive*** *if for some (equivalently every) $x_0 \in X$,*

$$d(x, x_0) \to \infty \quad \Longrightarrow \quad f(x) \to +\infty.$$

In a normed space this is usually written as $\|x\| \to \infty \Rightarrow f(x) \to +\infty$.

Example 2.3. *The function $f(x) = |x|$ on $\mathbb{R}$ is coercive. In contrast,*

$$f(x) = \begin{cases} x, & x \ge 0, \\ 0, & x < 0, \end{cases}$$

is not coercive because $f(x) = 0$ along $x \to -\infty$, even though it does have global minimizers (every $x \le 0$ is a minimizer).

Theorem 2.2 (Existence in finite dimension)**.** *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be proper, l.s.c., and coercive. Then f admits a global minimizer.*

Proof. Choose $\bar{x}$ such that $f(\bar{x}) < \infty$ and consider the sublevel set

$$C := \{x \in \mathbb{R}^d : f(x) \le f(\bar{x})\}.$$

Because f is l.s.c., C is closed. Coercivity implies that C is bounded; otherwise there would be $x_n \in C$ with $|x_n| \to \infty$, while $f(x_n) \le f(\bar{x})$, contradicting coercivity. Hence C is compact. Since $\inf_{\mathbb{R}^d} f = \inf_C f$, the previous theorem gives a minimizer. $\square$

Example 2.4 (Least-squares approximation)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The equation $Ax = b$ may have no solution, so one considers the least-squares problem*

$$\inf_{x\in\mathbb{R}^n} \|Ax - b\|^2.$$

Take a singular value decomposition

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m\times n}$ has diagonal entries

$$\sigma_1 \ge \cdots \ge \sigma_r > 0, \qquad \sigma_{r+1} = \sigma_{r+2} = \cdots = 0.$$

Here $r = \operatorname{rank}(A)$. Set

$$\widetilde{x} = V^T x, \qquad \widetilde{b} = U^T b.$$

Then

$$\|Ax-b\|^2 = \|\Sigma\widetilde{x}-\widetilde{b}\|^2 = \sum_{i=1}^{r} |\sigma_i\widetilde{x}_i - \widetilde{b}_i|^2 + \sum_{i=r+1}^{m} |\widetilde{b}_i|^2.$$

Thus every minimizer x satisfies

$$x = V\widetilde{x}, \qquad \widetilde{x}_i = \frac{\widetilde{b}_i}{\sigma_i}, \qquad i = 1,\dots,r,$$

while the components $\widetilde{x}_{r+1},\dots,\widetilde{x}_n$ are arbitrary. The minimizer is unique exactly when $\ker A = \{0\}$, equivalently when $r = n$. Moreover,

$$x \mapsto \|Ax-b\|^2 \text{ is coercive} \iff A \text{ is injective} \iff r = n,$$

which in particular requires $m \geq n$. Thus coercivity is sufficient but not necessary for existence.

2.4 Existence and non-existence in infinite dimensions

Sobolev spaces in one dimension

Definition 2.8. *For $1 \leq p < \infty$, $L^p(0,1)$ consists of (Lebesgue) measurable functions $f : (0,1) \to \mathbb{R}$ such that*

$$\int_0^1 |f|^p \, dx < \infty.$$

Functions equal almost everywhere are identified.

For $1 \leq p \leq \infty$, $L^p(0,1)$ is a Banach space with its usual norm. In particular, $L^2(0,1)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{L^2} := \int_0^1 f(t)g(t)\,dt.$$

Definition 2.9 (Weak derivative)**.** *Let $f \in L^1(0,1)$. A function $g \in L^1(0,1)$ is the* ***weak derivative*** *of f if*

$$\int_0^1 g\,\varphi\,dx = -\int_0^1 f\,\varphi'\,dx \qquad \forall \varphi \in C_c^\infty(0,1).$$

Remark 2.5. *If $f \in C^1(0,1)$, then the classical derivative f' is its weak derivative. Weak derivatives are unique up to equality almost everywhere.*

Example 2.5. *The function $f(x) = |x - \frac{1}{2}|$ is not classically differentiable at $x = \frac{1}{2}$, but it has weak derivative*

$$f'(x) = \begin{cases} -1, & x < \frac{1}{2}, \\ 1, & x > \frac{1}{2}, \end{cases} \qquad a.e.$$

Definition 2.10. *We define*

$$H^1(0,1) := \{f \in L^2(0,1) : f' \in L^2(0,1)\},$$

where f' denotes the weak derivative. With inner product

$$\langle f, g\rangle_{H^1} := \int_0^1 fg\,dt + \int_0^1 f'g'\,dt,$$

$H^1(0,1)$ is a Hilbert space.

Remark 2.6 (One-dimensional embedding)**.** *Every $f \in H^1(0,1)$ has a unique continuous representative. For this representative,*

$$f(x) - f(y) = \int_y^x f'(s)\,ds,$$

and hence, by Cauchy–Schwarz,

$$|f(x) - f(y)| \le |x - y|^{1/2}\|f'\|_{L^2(0,1)}.$$

Moreover, there is a constant $C > 0$ such that

$$\|f\|_\infty \le C\|f\|_{H^1}.$$

Thus $H^1(0,1)$ embeds continuously (indeed compactly) into $C([0,1])$.

Let us consider a concrete example that illustrates how Theorem 2.2 may fail in infinite dimension. We consider a minimization problem over the set of functions

$$\mathcal{A} := \{u \in H^1(0,1) : u(0) = u(1) = 1\},$$

where endpoint values are understood via the continuous representative. Specifically, we look for solutions of the following problem:

$$\inf_{u\in\mathcal{A}} J(u), \qquad J(u) = \int_0^1 \Big[(|u'(t)| - 1)^2 + u(t)^2\Big]dt.$$

Let us verify the assumtions of Theorem 2.2:

- ***Continuity.*** If $u_n \to u$ in H^1, then $u_n' \to u'$ in L^2 and $u_n \to u$ in L^2. Since

  $$(|a| - 1)^2 = a^2 - 2|a| + 1,$$

  we have

  $$\int_0^1 (|u_n'| - 1)^2\,dt \to \int_0^1 (|u'| - 1)^2\,dt,$$

  using L^2 convergence and the inequality $\big||a| - |b|\big| \le |a - b|$. Therefore J is strongly continuous on H^1.

- ***Coercivity.*** Using $2a \le \frac{1}{2}a^2 + 2$ for $a \ge 0$,

$$\begin{aligned} J(u) &= \|u'\|_{L^2}^2 - 2\|u'\|_{L^1} + 1 + \|u\|_{L^2}^2 \\ &\ge \frac{1}{2}\|u'\|_{L^2}^2 + \|u\|_{L^2}^2 - 1. \end{aligned}$$

  Hence J is coercive on H^1.

Nevertheless, no minimizer exists. To see this, define the 1-periodic tent function

$$u_1(t) = \frac{1}{2} - \left| t - \frac{1}{2} \right|, \qquad t \in [0,1],$$

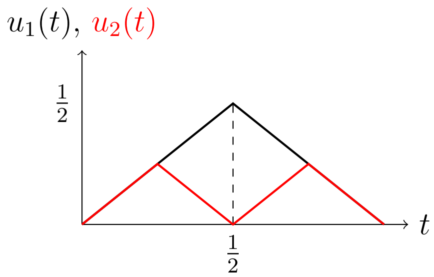
and extend it periodically to $\mathbb{R}$. For $n \in \mathbb{N}$, define

$$u_n(t) := \frac{1}{n} u_1(nt).$$

Then $u_n \in \mathcal{A}$, $|u_n'| = 1$ a.e., and moreover

$$J(u_n) = \int_0^1 u_n(t)^2 \, dt \longrightarrow 0.$$

Thus $\inf_{\mathcal{A}} J = 0$. But $J(u) = 0$ would force both $u \equiv 0$ and $|u'| = 1$ a.e., which is impossible; in particular $u \equiv 0$ violates $u(0) = u(1) = 1$.



Key problem: boundedness in an infinite-dimensional Hilbert space does not imply compactness in the strong topology. We therefore need a weaker topology to gain compactness and apply the direct method.

Weak convergence

Definition 2.11. *Let H be a Hilbert space. A sequence $(x_n)_n \subset H$ converges* ***weakly*** *to $x \in H$, written $x_n \rightharpoonup x$, if*

$$\langle x_n, y \rangle \to \langle x, y \rangle \qquad \forall y \in H.$$

Equivalently, $\ell(x_n) \to \ell(x)$ for every continuous linear functional $\ell \in H^*$.

Remark 2.7. *Weak limits are unique. If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then*

$$\langle x - y, z \rangle = 0 \qquad \forall z \in H,$$

so taking $z = x - y$ gives $x = y$.

Remark 2.8. *Strong convergence implies weak convergence. The sequence* (u_n) *above satisfies*

$$\|u_n\|_{L^2} \to 0,$$

so $u_n \to 0$ *strongly in* L^2 *and hence weakly in* L^2*. However,* $\|u_n'\|_{L^2} = 1$*, so* $u_n \not\to 0$ *strongly in* H^1*. In fact* $u_n \rightharpoonup 0$ *in* H^1*: the rapidly oscillating derivatives* u_n' *have zero mean over each period and therefore converge weakly to zero in* L^2*.*

Theorem 2.3 (Weak sequential compactness in Hilbert spaces)**.** *Let* H *be a Hilbert space and let* $(x_n)_n$ *be bounded in* H*. Then there exists a subsequence* x_{n_k} *and* $x \in H$ *such that*

$$x_{n_k} \rightharpoonup x.$$

In a separable Hilbert space this follows from the weak compactness of closed balls together with metrizability of the weak topology on bounded sets. More generally, Hilbert spaces are reflexive, so bounded sequences are weakly relatively compact.

Example 2.6 (A direct-method argument)**.** *Let* $F \in C([a,b] \times \mathbb{R})$ *and assume that for some* $C > 0$*,*

$$F(t,x) \geq C(|x|^2 - 1) \qquad \forall (t,x) \in [a,b] \times \mathbb{R}.$$

Consider

$$\inf_{\substack{u \in H^1(a,b) \\ u(a)=0,\ u(b)=1}} J(u), \qquad J(u) = \int_a^b \Big(F(t,u(t)) + |u'(t)|^2 \Big) dt.$$

Let

$$\bar{u}(t) = \frac{t-a}{b-a}$$

be an admissible competitor, and let $(u_n)_n$ *be a minimizing sequence with* $J(u_n) \leq J(\bar{u}) + 1$ *for all* n*.*

1. ***Uniform*** H^1 ***bound.*** *From the lower bound on* F*,*

   $$J(u_n) \geq C\|u_n\|^2_{L^2(a,b)} - C(b-a) + \|u_n'\|^2_{L^2(a,b)}.$$

   Hence $(u_n)_n$ *is bounded in* $H^1(a,b)$*.*

2. ***Weakly convergent subsequence.*** *By weak sequential compactness, after passing to a subsequence,*

   $$u_n \rightharpoonup u \qquad \text{in } H^1(a,b).$$

   In particular, $u_n' \rightharpoonup u'$ *in* $L^2(a,b)$*.*

3. ***Recover the boundary conditions.*** *A bounded sequence in* $H^1(a,b)$ *is uniformly bounded and equicontinuous by the one-dimensional estimate above. By Arzelà–Ascoli, after passing to a further subsequence,*

   $$u_n \to u \qquad \text{uniformly on } [a,b].$$

Therefore

$$u(a) = 0, \qquad u(b) = 1.$$

The uniform limit agrees with the weak limit by uniqueness.

4. ***Lower semicontinuity.*** *Since $u_n' \rightharpoonup u'$ in L^2,*

$$\int_a^b |u'|^2 \, dt \leq \liminf_{n\to\infty} \int_a^b |u_n'|^2 \, dt.$$

Moreover, uniform convergence implies that u_n stays in a common compact subset of $\mathbb{R}$, and the continuity of F gives

$$F(t, u_n(t)) \to F(t, u(t))$$

uniformly on $[a, b]$. Hence

$$\int_a^b F(t, u_n(t)) \, dt \to \int_a^b F(t, u(t)) \, dt.$$

Combining the two estimates yields

$$J(u) \leq \liminf_{n\to\infty} J(u_n) = \inf J.$$

Thus u is a minimizer.

Example 2.7 (Failure of weak lower semicontinuity)**.** *For the non-existence example above,*

$$J_1(u) := \int_0^1 (|u'| - 1)^2 \, dt$$

is strongly continuous but not weakly lower semicontinuous on H^1. Indeed $u_n \rightharpoonup 0$ in H^1 while

$$J_1(u_n) = 0, \qquad J_1(0) = 1.$$

Thus

$$J_1(0) > \liminf_{n\to\infty} J_1(u_n) = 0.$$

Remark 2.9. *Weak lower semicontinuity is stronger than strong lower semicontinuity: every strongly convergent sequence is weakly convergent, so a functional that is weakly l.s.c. is automatically strongly l.s.c..*

Exercise 2.1. *Let*

$$J(u) = \int_0^1 f(u(t)) \, dt, \qquad u \in L^2(0, 1),$$

where $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$. Show that if J is weakly l.s.c. on $L^2(0, 1)$, then f must be convex and l.s.c. Also give an example of a non-convex f for which J is nevertheless strongly l.s.c..